

THE SOH OPERATOR SYSTEM

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ABSTRACT. In this paper we examine a natural operator system structure on Pisier’s self-dual operator space. We prove that this operator system is completely order isomorphic to its dual with the cb-condition number of this isomorphism as small as possible. We examine further properties of this operator system and use it to create a new tensor product on operator systems.

1. INTRODUCTION

Pisier[13] proved that, for each dimension, there is a unique operator space with the property that it is completely isometrically isomorphic to its dual space. In this paper we study the analogous problem in the matrix ordered setting. Since the dual of a matrix ordered space is still a matrix ordered space, it is natural to ask if a matrix ordered space is completely order isomorphic to its dual.

Unlike the operator space case, there are many operator systems that are completely order isomorphic to their matrix-ordered dual. Since the dual of an operator system also carries a matrix norm, it is natural to ask if an operator system is ever simultaneously completely order isomorphic and completely norm isomorphic to its dual. We will show that this is impossible. In fact, we will prove that any complete order isomorphism between an operator system and its dual has a cb-condition number that is bounded below by 2.

We will see that for the many standard examples of finite dimensional operator systems that are completely order isomorphic to their duals, the cb-condition number of this order isomorphism grows unbounded as the dimension tends to infinity.

We will then create a “natural” operator system from Pisier’s $OH(n)$ spaces, that we denote by $SOH(n)$ and show that these operator systems have the property that there exists a map from the space to its dual that is a complete order isomorphism and has cb-condition number of exactly 2.

We then explore some further properties and applications of the operator systems $SOH(n)$. We prove that subsystems and quotients of $SOH(n)$ are completely order isomorphic to $SOH(m)$ for some $m \leq n$.

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Finally, we use “approximate cp-factorization through SOH” to create a new tensor product on operator systems and examine some of its properties.

2. OPERATOR SYSTEM AND OPERATOR SPACE DUALITY

We assume that the reader is familiar with the basic definitions and properties of operator spaces, operator systems, completely bounded and completely positive maps. For more details the reader should see the books [10, 12]. We only review the basic definitions of duals of operator spaces and operator systems, since these are the objects that we wish to contrast.

If V is an operator space, then the space of bounded linear functionals on V , denoted V^d , comes equipped with a natural **dual matrix-norm**. Briefly, a matrix of linear functionals $F = (f_{i,j}) \in M_n(V^d)$ is identified with a linear map $F : V \rightarrow M_n$ and we set $\|(f_{i,j})\|_n = \|F\|_{cb}$.

Recall that given a $*$ -vector space V , the vector space $M_n(V)$ is also a $*$ -vector space with $*$ -operation given by $(v_{i,j})^* = (v_{i,j}^*)^t$ where t denotes the transpose. By a **matrix order** on V we mean a family of cones of self-adjoint elements, $C_n \subseteq M_n(V)_h$, that satisfy:

- (1) $C_n \cap (-C_n) = \{0\}$,
- (2) $M_n(V)$ is the complex span of C_n ,
- (3) if $A = (a_{i,j}) \in M_{n,m}$ is a matrix of scalars and $(v_{i,j}) \in C_n$, then $A^*(v_{i,j})A = (\sum_{k,l} \overline{a_{i,k}} v_{k,l} a_{l,j}) \in C_m$.

We call such a $*$ -vector space a **matrix-ordered space** and simplify notation, when possible, by setting $C_n = M_n(V)^+$. Note that if $V_1 \subseteq V$ is a $*$ -invariant vector subspace, then the cones $C_n \cap M_n(V_1)$ endow V_1 with a matrix-order that we call the **subspace order**, or more simply, we refer to $V_1 \subseteq V$ as the **matrix ordered subspace**.

Given two matrix ordered spaces V and W we call a map $\phi : V \rightarrow W$ **completely positive** provided that $\phi^{(n)} : M_n(V) \rightarrow M_n(W)$ is positive for all n .

Given a matrix-ordered space V , we let V^\dagger denote the vector space of all linear functionals on V . Given a linear functional $f : V \rightarrow \mathbb{C}$, if we let $f^* : V \rightarrow \mathbb{C}$ be the linear functional $f^*(v) = \overline{f(v^*)}$, then this makes V^\dagger a $*$ -vector space. We identify an $n \times n$ matrix of linear functionals $(f_{i,j})$ with the linear map, $F : V \rightarrow M_n$ defined by $F(v) = (f_{i,j}(v))$, and set $M_n(V^\dagger)^+$ equal to the cone of completely positive maps. Then this gives a sequence of cones on the dual that satisfy properties (1) and (3), but not generally (2). When V is also a normed space, then we let V^d denote the space of bounded linear functionals on V , which is a subspace of V^\dagger and is endowed with the subspace order.

However, when V is an operator system, then V^d endowed with this set of cones is a matrix-ordered space and we refer to this as the **matrix-ordered dual of V** .

The easiest way to see that these cones span, is to use Wittstock's decomposition theorem [14, 10] which says that the completely bounded maps on an operator system are the complex span of the completely positive maps.

Since every operator system V is also an operator space, its dual comes equipped with two structures, an operator space structure and a matrix-order structure. We wish to focus on the contrast between these two structures.

We begin with some examples. We always identify the dual of \mathbb{C}^n with \mathbb{C}^n again via the map that sends the standard basis $\{e_j\}$ to the dual basis $\{\delta_j\}$.

Example 2.1. *The identification of ℓ_n^∞ with the continuous functions on an n point space makes ℓ_n^∞ into an operator system with $\sum_j A_j \otimes e_j \in M_m(\ell_n^\infty)^+$ iff $A_j \in M_m^+$ for all j . Moreover, a map $\Phi : \ell_n^\infty \rightarrow M_m$ with $\Phi(e_j) = A_j$ is completely positive iff $A_j \in M_m^+$ for all j . From this it follows that the map $e_j \rightarrow \delta_j$ is a complete order isomorphism between ℓ_n^∞ and $(\ell_n^\infty)^d$. Thus, as a matrix-ordered space ℓ_n^∞ is self-dual.*

On the other hand ℓ_n^∞ is also an operator space and the normed dual is ℓ_n^1 via the same identification. The operator space structure on $(\ell_n^\infty)^d$ is the operator space $MAX(\ell_n^1) = \text{span}\{u_1, \dots, u_n\} \subseteq C^(\mathbb{F}_n)$ where $C^*(\mathbb{F}_n)$ denotes the full C^* -algebra of the free group on n generators and u_j are the generators [15]. In this case the norm and cb -norm of the identity map $id : \ell_n^\infty \rightarrow \ell_n^1$ is n . The cb -condition number is $\|id\|_{cb}\|id^{-1}\|_{cb} = n$.*

Example 2.2. *If we consider M_n as an operator system with the usual structure, then [11] the map that sends the matrix units $E_{i,j}$ to their dual basis $\{\delta_{i,j}\}$ defines a complete order isomorphism between M_n and M_n^d . This map sends the identity operator $I_n = \sum_{j=1}^n E_{j,j}$ to the trace functional Tr , where $\text{Tr}((a_{i,j})) = \sum_{j=1}^n a_{j,j}$. Thus, M_n is also completely order isomorphic to its dual.*

However, recall that the normed dual, with this same identification is the trace class matrices S_n^1 , together with their operator space structure. Again the norm, cb -norm, and cb -condition number of the identity map (between these n^2 dimensional spaces) is n .

Thus, in both these examples we have operator systems that are completely order isomorphic to their ordered duals, but the identification does not preserve the operator space structure of the dual.

3. THE OPERATOR SYSTEM $\text{SOH}(N)$

In this section, for each cardinal number n , we introduce an operator system $\text{SOH}(n)$ of dimension $n + 1$ based on Pisier's self-dual operator space $\text{OH}(n)$ and analyze their properties. In particular, we prove that these operator systems are self-dual as matrix-ordered spaces and that the natural map from $\phi : \text{SOH}(n) \rightarrow \text{SOH}(n)^d$ satisfies $\|\phi\|_{cb} \cdot \|\phi^{-1}\|_{cb} = 2$,

which we show is as close to being a complete isometry as is possible for any operator system that is completely order isomorphic to its dual.

We begin with a result that shows that the lower bound of 2 is sharp.

Proposition 3.1. *Let \mathcal{S} be an operator system of dimension at least 2 and assume that $\phi : \mathcal{S} \rightarrow \mathcal{S}^d$ is a complete order isomorphism of \mathcal{S} onto its dual space. Then $\|\phi\| \cdot \|\phi^{-1}\| \geq 2$.*

Proof. Let I denote the identity element of \mathcal{S} and let $\delta_0 = \phi(I)$. Choose $H = H^* \in \mathcal{S}$ that is not in the span of I . Since δ_0 is positive, $\delta_0(H) \in \mathbb{R}$. Replacing H by $H - \delta_0(H)I$ we may assume that $\delta_0(H) = 0$. Now let $\delta_1 = \phi(H)$, which is a self-adjoint functional on \mathcal{S} . Set $M = \inf\{r : rI \geq H\}$ and set $m = \sup\{rI : H \geq rI\}$. Since H is not a multiple of I , it follows that $m < M$. For any real numbers a, b we will have that $\|aI + bH\| = \max\{|a + bM|, |a + bm|\}$ and that $aI + bH \geq 0$ iff $\min\{a + bM, a + bm\} \geq 0$. Since ϕ is a complete order isomorphism, $a\delta_0 + b\delta_1$ is completely positive iff $\min\{a + bM, a + bm\} \geq 0$.

Now note that $\|MI - H\| = M - m = \|H - mI\|$ and that $MI - H \geq 0$, $H - mI \geq 0$, and so $M\delta_0 - \delta_1$ and $\delta_1 - m\delta_0$ are both completely positive. Let $\delta_1(I) = s$. The complete positivity of these last two maps, implies that $\|M\delta_0 - \delta_1\| = (M\delta_0 - \delta_1)(I) = M - s \geq 0$ and that $\|\delta_1 - m\delta_0\| = (\delta_1 - m\delta_0)(I) = s - m \geq 0$. Hence, $m \leq s \leq M$.

Finally,

$$\|\phi\| \cdot \|\phi^{-1}\| \geq \max\left\{\frac{\|MI - H\|}{\|M\delta_0 - \delta_1\|}, \frac{\|H - mI\|}{\|\delta_1 - m\delta_0\|}\right\} = \max\left\{\frac{M - m}{M - s}, \frac{M - m}{s - m}\right\} \geq 2.$$

This last inequality follows by observing that the minimum of this maximum over s occurs when $s = (M + m)/2$. \square

To construct SOH , we consider the finite dimensional case, the extension to infinite dimensions is standard. We use a few facts that are implicitly contained in Pisier[12, Exercise 7.2]. Fix a Hilbert space of dimension n and let $\{e_i\}$ be an orthonormal basis. Assume that $OH(n) \subseteq B(\mathcal{H})$ is a completely isometric inclusion, so that e_i are identified with operators. Let

$$H_i = \begin{pmatrix} 0 & e_i \\ e_i^* & 0 \end{pmatrix} \in B(\mathcal{H} \oplus \mathcal{H}),$$

so that the H_i 's are self-adjoint operators.

Given matrices, we have that

$$\begin{aligned} \left\| \sum_i A_i \otimes H_i \right\| &= \max\left\{ \left\| \sum_i A_i \otimes e_i \right\|, \left\| \sum_i A_i \otimes e_i^* \right\| \right\} = \\ &= \max\left\{ \left\| \sum_i A_i \otimes \overline{A_i} \right\|^{1/2}, \left\| \sum_i A_i^* \otimes A_i^t \right\|^{1/2} \right\} = \left\| \sum_i A_i \otimes e_i \right\|. \end{aligned}$$

This last equality follows since $A^t \otimes B^t = (A \otimes B)^t$ and so,

$$\left\| \sum_i A_i^* \otimes A_i^t \right\| = \left\| \left(\sum_i \overline{A_i} \otimes A_i \right)^t \right\| = \left\| \sum_i \overline{A_i} \otimes A_i \right\| = \left\| \sum_i A_i \otimes e_i \right\|^2.$$

Note in particular, we have that $\|\sum_i A_i \otimes e_i\| = \|\sum_i A_i^* \otimes e_i\| = \|\sum_i A_i^t \otimes e_i\|$.

Thus, the map $e_i \rightarrow H_i$ is a complete isometry and we have that $OH(n)$ is also the span of these self-adjoint elements. The particular form of these self-adjoint operators will be useful in the sequel.

For notational convenience we let H_0 denote the identity operator on $\mathcal{H} \oplus \mathcal{H}$.

Definition 3.2. We let $SOH(n) \subseteq B(\mathcal{H} \oplus \mathcal{H})$ denote the $(n+1)$ -dimensional operator system that is the span of the set $\{H_i : 0 \leq i \leq n\}$.

We now examine the norm and order structure on $SOH(n)$.

Proposition 3.3. Let $A_i \in M_m, 0 \leq i \leq n$. Then the following are equivalent:

- $\sum_{i=0}^n A_i \otimes H_i$ is positive,
- $A_0 \otimes H_0 - \sum_{i=1}^n A_i \otimes H_i$ is positive,
- $A_0 \in M_m^+, A_i = A_i^*, 1 \leq i \leq n$ and $-A_0 \otimes \overline{A_0} \leq \sum_{i=1}^n A_i \otimes \overline{A_i} \leq +A_0 \otimes \overline{A_0}$, in $M_m \otimes M_m = M_{m^2}$.

Proof. Let $U = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix} \in B(\mathcal{H} \oplus \mathcal{H})$, which is unitary. Note that $U^* H_0 U = H_0$ and $U^* H_i U = -H_i, 1 \leq i \leq n$, from which the equivalence of the first two statements follows.

Adding the first two equations shows that $A_0 \geq 0$. Since a positive element must be self-adjoint it follows that $A_i = A_i^*, 1 \leq i \leq n$.

To see the final equations, first assume that A_0 is positive and invertible. Then $\sum_{i=0}^n A_i \otimes H_i$ is positive iff $(A_0 \otimes H_0)^{-1/2} (\sum_{i=0}^n A_i \otimes H_i) (A_0 \otimes H_0)^{-1/2}$ is positive which is iff $I_m \otimes H_0 + \sum_{i=1}^n B_i \otimes H_i$ is positive, where $B_i = A_0^{-1/2} A_i A_0^{-1/2}$. As operators on $\mathcal{H} \oplus \mathcal{H}$, we have that

$$\begin{pmatrix} I_{\mathcal{H}} & \sum_i B_i \otimes e_i \\ \sum_i B_i \otimes e_i & I_{\mathcal{H}} \end{pmatrix}$$

is positive.

This last equation is equivalent to requiring that the $(1,2)$ -entry of this operator matrix is a contraction and hence, $\|\sum_i B_i \otimes \overline{B_i}\| \leq 1$. But since these matrices are self-adjoint, this is equivalent to

$$-I_m \otimes I_m \leq \sum_i B_i \otimes \overline{B_i} \leq I_m \otimes I_m.$$

Conjugating this last result by $A_0^{1/2} \otimes \overline{A_0^{1/2}}$ yields the desired inequality.

When A_0 is not invertible, one first considers $A_0 + rI_m, r > 0$ and then lets $r \rightarrow 0$. This completes the proof. \square

We now consider the matrix-ordered dual of $SOH(n)$. To this end we let $\delta_i \in SOH(n)^d, 0 \leq i \leq n$ denote the linear functionals that satisfy, $\delta_i(H_j) = \delta_{i,j}, 0 \leq i, j \leq n$.

Theorem 3.4. *The map $\kappa : SOH(n) \rightarrow SOH(n)^d$ defined by $\kappa(H_i) = \delta_i$, $0 \leq i \leq n$, is a complete order isomorphism that satisfies*

$$\left\| \sum_{i=0}^n A_i \otimes \delta_i \right\|_{cb} \leq \left\| \sum_{i=0}^n A_i \otimes H_i \right\| \leq 2 \left\| \sum_{i=0}^n A_i \otimes \delta_i \right\|_{cb}$$

for any matrices $A_0, \dots, A_n \in M_m$ and any m and $\|\kappa\|_{cb} \cdot \|\kappa^{-1}\|_{cb} = 2$.

Proof. First, we prove that κ is completely positive. Keeping the notation from the last proof, assume that $\sum_{i=0}^n A_i \otimes H_i$ is positive. We must prove that the map $\Phi : SOH(n) \rightarrow M_m$ given by $\Phi(X) = \sum_{i=0}^n A_i \otimes \delta_i(X)$ is completely positive. Assume that A_0 is invertible and define B_i as above. Let $P = \sum_{i=0}^n P_i \otimes H_i \in M_q(SOH(n))^+$. We must show that

$$\Phi^{(q)}(P) = \sum_{i=0}^n A_i \otimes P_i \in (M_n \otimes M_q)^+.$$

Assuming that P_0 is also invertible, we set $Q_i = P_0^{-1/2} P_i P_0^{-1/2}$. By the last Proposition, we have that $\|\sum_{i=1}^n B_i \otimes e_i\| \leq 1$ and $\|\sum_{i=1}^n Q_i \otimes e_i\| \leq 1$. Hence, by the self-duality of $OH(n)$, we have that $\|\sum_{i=1}^n B_i \otimes Q_i\|_{M_m \otimes M_q} \leq 1$. Using the fact that all these matrices are self-adjoint, yields

$$-I_m \otimes I_q \leq \sum_{i=1}^n B_i \otimes Q_i \leq +I_m \otimes I_q.$$

Thus, $I_m \otimes I_q + \sum_{i=1}^n B_i \otimes Q_i \geq 0$, which after conjugation by $A_0^{1/2} \otimes P_0^{1/2}$ yields that $\Phi^{(q)}(P) \geq 0$.

Conversely, if $\Phi = \sum_{i=0}^n A_i \otimes \delta_i \in M_m(SOH(n)^d)$ is completely positive, then it follows that $A_0 \geq 0$, and that $A_i = A_i^*$, $1 \leq i \leq n$. Taking B_i 's as before, we have that $\Psi = I_m \otimes \delta_0 + \sum_{i=1}^n B_i \otimes \delta_i$ is a unital completely positive map and hence completely contractive. Applying this map to any element $\sum_i C_i \otimes e_i \in M_q(OH(n))$ of norm less than one, yields that $\|\sum_{i=1}^n B_i \otimes C_i\| \leq 1$. Thus, by self-duality of $OH(n)$ we have that $\|\sum_{i=1}^n B_i \otimes \overline{B_i}\| \leq 1$. Hence, $-I_m \otimes I_m \leq \sum_{i=1}^n B_i \otimes \overline{B_i} \leq +I_m \otimes I_m$ and the Proposition 3.3 implies that $\sum_{i=0}^n A_i \otimes H_i$ is positive.

Thus, κ is a complete order isomorphism.

We now consider the norm inequalities. Let $X = \sum_{i=0}^n A_i \otimes H_i$, set $\Phi = \sum_{i=0}^n A_i \otimes \delta_i$ and assume that $\|X\|_{SOH(n)} \leq 1$. Here, the matrices A_i are no longer necessarily self-adjoint. We then have that

$$0 \leq \begin{pmatrix} I_{\mathcal{H}} \otimes I_m & X \\ X^* & I_{\mathcal{H}} \otimes I_m \end{pmatrix} = \begin{pmatrix} I_m & A_0 \\ A_0^* & I_m \end{pmatrix} \otimes H_0 + \sum_{i=1}^n \begin{pmatrix} 0 & A_i \\ A_i^* & 0 \end{pmatrix} \otimes H_i.$$

From the fact that κ is completely positive, it follows that

$$\begin{pmatrix} I_m & A_0 \\ A_0^* & I_m \end{pmatrix} \otimes \delta_0 + \sum_{i=1}^n \begin{pmatrix} 0 & A_i \\ A_i^* & 0 \end{pmatrix} \otimes \delta_i = \begin{pmatrix} I_m \otimes \delta_0 & \sum_{i=0}^n A_i \otimes \delta_i \\ \sum_{i=0}^n A_i^* \otimes \delta_i & I_m \otimes \delta_0 \end{pmatrix} = \begin{pmatrix} \Psi & \Phi \\ \Phi^* & \Psi \end{pmatrix},$$

and Ψ is a unital completely positive map. Hence, $\|\Phi\|_{cb} \leq 1$ and it follows that $\|\kappa^{(m)}(X)\|_{cb} \leq \|X\|$ for any $X \in M_m(SOH(n))$ and any m .

Conversely, assume that $\Phi = \sum_{i=0}^n A_i \otimes \delta_i$. To prove the other inequality, it will be enough to assume that $\|\Phi\|_{cb} \leq 1$ and show that $\|X\|_{SOH(n)} \leq 2$.

Since $\|\Phi\|_{cb} \leq 1$, there exist unital completely positive maps $\Psi_j : SOH(n) \rightarrow$

$M_m, j = 1, 2$ such that the map $\Gamma = \begin{pmatrix} \Psi_1 & \Phi \\ \Phi^* & \Psi_2 \end{pmatrix} : SOH(n) \rightarrow M_{2m}$

is completely positive. Writing $\Psi_j = \sum_{i=0}^n C_i^j \otimes \delta_i$, we have that $\Gamma = \sum_{i=0}^n \begin{pmatrix} C_i^1 & A_i \\ A_i^* & C_i^2 \end{pmatrix} \otimes \delta_i$. Moreover, since the maps Ψ_j are unital, $C_0^1 = C_0^2 = I_m$. By the Proposition and the fact that κ is a complete order isomorphism, we know that the fact that Γ is completely positive implies that $\Gamma_1 = \begin{pmatrix} I_m & A_0 \\ A_0^* & I_m \end{pmatrix} \otimes \delta_0 - \sum_{i=1}^n \begin{pmatrix} C_i^1 & A_i \\ A_i^* & C_i^2 \end{pmatrix} \otimes \delta_i$ is completely positive. Adding $\Gamma + \Gamma_1$, and using the positivity, yields that $\|A_0\| \leq 1$.

Next, if we let Γ_2 be the completely positive map that we get by conjugating the coefficients of Γ_1 by the unitary $U = \begin{pmatrix} -I_m & 0 \\ 0 & I_m \end{pmatrix}$, we find

that $\Gamma_2 = \begin{pmatrix} I_m & -A_0 \\ -A_0^* & I_m \end{pmatrix} \otimes \delta_0 + \sum_{i=1}^n \begin{pmatrix} -C_i^1 & A_i \\ A_i^* & -C_i^2 \end{pmatrix} \otimes \delta_i$. The average $1/2(\Gamma + \Gamma_2) = \begin{pmatrix} I_m & 0 \\ 0 & I_m \end{pmatrix} \otimes \delta_0 + \sum_{i=1}^n \begin{pmatrix} 0 & A_i \\ A_i^* & 0 \end{pmatrix} \otimes \delta_i$ is a unital completely positive map.

Using that κ is a complete order isomorphism and replacing the δ_i 's by H_i 's, yields that $\|\sum_{i=1}^n A_i \otimes H_i\| \leq 1$. Hence,

$$\left\| \sum_{i=0}^n A_i \otimes H_i \right\| \leq \|A_0 \otimes H_0\| + \left\| \sum_{i=1}^n A_i \otimes H_i \right\| \leq 2$$

and the desired inequality follows.

Finally, we have that $\|\kappa\|_{cb} \leq 1$ and $\|\kappa^{-1}\|_{cb} \leq 2$, so that $\|\kappa\| \cdot \|\kappa^{-1}\|_{cb} \leq 2$ and so we must have equality by Proposition 3.1. \square

Remark 3.5. *By the above results we see that, among all self-dual operator systems, the operator systems $SOH(n)$ achieve the minimal cb-condition number of 2. However, this does not uniquely characterize these spaces. In fact, M_2 is another self-dual operator system that attains this minimum. One other example is ℓ_2^∞ , but it is not hard to see that this operator system is unital, completely order isomorphic to $SOH(1)$. It would be interesting to try and characterize the self-dual operator systems that attain this minimal cb-condition number.*

4. SOME STRUCTURE RESULTS OF $SOH(N)$

In [12], $OH(n)$ is defined in a basis-free fashion. In this section we show that $SOH(n)$ is also independent of basis, which leads to proving every

quotient and operator subsystem of $SOH(n)$ is unittally completely order isomorphic to some $SOH(m)$. We also derive a few properties of $SOH(n)$ that will be useful in the later sections. To avoid ambiguity, whenever we work with $SOH(n)$ and $SOH(m)$, we denote $H_i^{(n)}$ and $H_j^{(m)}$, respectively, the basis elements H_i as given in section 3.

Proposition 4.1. *Let $1 \leq n \leq m$ and let $\{\vec{u}_i = (u_{ij}) \in \mathbb{R}^m\}_{i=1}^n$ be an orthonormal set. Then the map $\Phi: SOH(n) \rightarrow SOH(m)$ defined by $\Phi(I) = I$ and $\Phi(H_i^{(n)}) := \sum_{j=1}^m u_{ij} H_j^{(m)}$ is a complete order inclusion.*

Proof. Consider $\sum_{i=0}^n A_i \otimes H_i^{(n)} \in M_p \otimes SOH(n)$. Let $B_0 = A_0$ and for $j = 1, \dots, n$, let $B_j = \sum_{i=1}^n u_{ij} A_i$. Then $\sum_{i=0}^n A_i \otimes \Phi(H_i)$ is

$$B_0 \otimes H_0^{(m)} + \sum_{i=1}^n A_i \otimes \left(\sum_{j=1}^m u_{ij} H_j^{(m)} \right) = B_0 \otimes I + \sum_{j=1}^m B_j \otimes H_j^{(m)}.$$

It is obvious that $B_j = B_j^*$; and by orthonormality of the \vec{u}_i 's,

$$\begin{aligned} \sum_{j=1}^m B_j \otimes \overline{B_j} &= \sum_{j=1}^m \left(\sum_{i,k=1}^n u_{ij} u_{kj} \right) A_i \otimes \overline{A_k} = \sum_{i,k=1}^n \left(\sum_{j=1}^m u_{ij} u_{kj} \right) A_i \otimes \overline{A_k} \\ &= \sum_{i,k=1}^n \delta_{i,k} A_i \otimes \overline{A_k} = \sum_{i=1}^n A_i \otimes \overline{A_i}. \end{aligned}$$

Therefore, $\{A_i\}_{i=0}^n$ satisfies the third condition in Proposition 3.3 if and only if $\{B_j\}_{j=0}^m$ satisfies the same condition, proving that $\sum_{i=0}^n A_i \otimes H_i^{(n)} \geq 0$ if and only if $\sum_{i=0}^n A_i \otimes \Phi(H_i^{(n)}) \geq 0$; this is equivalent to Φ being a unital complete order inclusion. \square

Corollary 4.2. *Let $U = [u_{ij}] \in M_n(\mathbb{R})$ be an orthonormal matrix and set $K_0 = H_0$, $K_i = \sum_{j=1}^n u_{ij} H_j$. Then the map $\Phi: SOH(n) \rightarrow SOH(n)$ given by $\Phi(H_0) = K_0$ and $\Phi(H_i) = K_i$ is a unital complete order isomorphism.*

Given $n \leq m$, it is now clear that $SOH(n) \subset_{uoi} SOH(m)$. We will see that every operator subsystem of $SOH(m)$ is necessarily $SOH(n)$.

Corollary 4.3. *If \mathcal{T} is a operator subsystem of $SOH(m)$ of dimension $n+1$, then \mathcal{T} is unittally completely order isomorphic to $SOH(n)$.*

Proof. Let $\{K_0 = I, K_i = K_i^*: i = 1, \dots, n\}$ be a basis for \mathcal{T} . Without loss of generality, we assume for each $i = 1, \dots, n$, $K_i = \sum_{j=1}^m a_{ij} H_j^{(m)}$ for some $a_{ij} \in \mathbb{R}$. We first claim that the vectors $\vec{a}_i = (a_{ij}) \in \mathbb{R}^m$ are linearly independent. For if not, then $\vec{a}_i = \sum_{k=1, k \neq i}^n \lambda_k \vec{a}_k$, for some i , leading to $K_i = \sum_{j=1}^m \sum_{k=1, k \neq i}^n \lambda_k H_j^{(m)}$, which contradicts our assumption.

Now consider the n -dimensional subspace of \mathbb{R}^m spanned these \vec{a}_i 's. Pick an orthonormal basis $\{\vec{u}_i = (u_{ij}) \in \mathbb{R}^m\}_{i=1}^n$ for this subspace and define $\Phi: SOH(n) \rightarrow SOH(m)$ by $\Phi(I) = I$ and $\Phi(H_i^{(n)}) = \sum_{j=1}^m u_{ij} H_j^{(m)}$. By

the last proposition, Φ is a complete order inclusion. It remains to check that the image of Φ is \mathcal{T} . Since every $\vec{a}_i = \sum_{k=1}^n \lambda_k^i \vec{u}_k$, for each K_i we can write

$$K_i = \sum_{j=1}^m a_{ij} H_j^{(m)} = \sum_{j=1}^m \sum_{k=1}^n \lambda_k^i u_{kj} H_j^{(m)} = \sum_{k=1}^n \lambda_k^i \Phi(H_j^{(n)}),$$

proving that $\Phi(SOH(n)) = \mathcal{T}$. Consequently $\mathcal{T} \cong_{ucoi} SOH(n)$ via Φ . \square

Hence every operator subsystem of $SOH(n)$ is again of the same form. The next result characterizes quotients of $SOH(n)$ based on self-duality.

Proposition 4.4. *Let \mathcal{J} be a non-trivial self-adjoint subspace of $SOH(n)$. Then the following are equivalent:*

- (1) \mathcal{J} is the kernel of some unital, completely positive map with domain $SOH(n)$.
- (2) There exist $m < n$ and a surjective unital completely positive map $\phi: SOH(n) \rightarrow SOH(m)$ such that $\mathcal{J} = \ker(\phi)$.
- (3) There is unital completely positive map ϕ on $SOH(n)$ for which $\mathcal{J} = \ker(\phi)$.

Proof. The direction (2) \implies (3) \implies (1) is obvious. Now assume (1) and let $q: SOH(n) \rightarrow SOH(n)/\mathcal{J}$ be the canonical quotient map. Then $q^d: (SOH(n)/\mathcal{J})^d \rightarrow SOH(n)^d = SOH(n)$ is a unital complete order embedding [2]. Since \mathcal{J} is non-trivial, $(SOH(n)/\mathcal{J})^d$ has dimension $m < n$ and by the last corollary $(SOH(n)/\mathcal{J})^d \cong SOH(m)$. By duality, $SOH(n)/\mathcal{J} \cong SOH(m)^d = SOH(m)$. \square

In [6, Section 8], it is shown that the coproduct of two operator systems \mathcal{S} and \mathcal{T} can be obtained by operator system quotients. Namely, $\mathcal{S} \oplus_1 \mathcal{T} \cong_{ucoi} (\mathcal{S} \oplus \mathcal{T})/\mathcal{J}$, where $\mathcal{J} = \mathbb{C}(1_{\mathcal{S}}, -1_{\mathcal{T}})$.

Proposition 4.5. *For any $p \in \mathbb{N}$, let $H_0^{(p)}, \dots, H_p^{(p)}$ denote the canonical basis for $SOH(p)$. Then for any $n, m \in \mathbb{N}$, the map $\phi: SOH(n) \oplus SOH(m) \rightarrow SOH(n+m)$ defined by $\phi(H_j^{(n)}) = H_j^{(n+m)}, 0 \leq j \leq n$*

and $\phi(H_j^{(m)}) = \begin{cases} H_0^{(n+m)}, & j = 0 \\ H_{n+j}^{(n+m)}, & j > 0 \end{cases}$ induces a unital completely positive map

$\Phi: SOH(n) \oplus_1 SOH(m) \rightarrow SOH(n+m)$, but this map is not an order isomorphism.

Proof. It is easily checked that the restriction of ϕ to each direct summand is a unital completely positive map. Hence, Φ is a unital completely positive map by the universal properties of the coproduct.

To see that Φ is not an order isomorphism, it suffices to show that $SOH(1) \oplus_1 SOH(1) \neq SOH(2)$. Suppose the contrary and consider the positive element $P = \sqrt{2}H_0^{(2)} + H_1^{(2)} + H_2^{(2)}$ in $SOH(2)$. Then there must be positive numbers a and b such that $(aH_0^{(1)} + H_1^{(1)})$ and $(bH_0^{(1)} + H_1^{(1)})$

are positive in $SOH(1)$ and sum to P in $SOH(2)$. By Proposition 3.3, each of a^2 and b^2 is greater than 1; however $a + b = \sqrt{2}$ implies that $2ab \leq 0$, contradicting a and b are positive. \square

Remark 4.6. *In an earlier version of this paper, we erroneously claimed that Φ was a complete order isomorphism. We would like to thank Ali S. Kavruk for pointing out this error.*

Proposition 4.7. *Let \mathcal{S} be an operator system and $\{h_i: h_i = h_i^*, \|h_i\| \leq 1\}_{i=1}^n \subset \mathcal{S}$. Then there is $r > 0$ such that the map $\phi: SOH(n) \rightarrow \mathcal{S}$ given by $H_0 \mapsto r1_{\mathcal{S}}$, $H_i \mapsto h_i$ is completely positive.*

Proof. Choose $r > n^{1/2}$ and suppose $A_0 \otimes H_0 + \sum_{i=1}^n A_i \otimes H_i$ is positive in $M_m \otimes SOH(n)$. We will show that $rA_0 \otimes 1_{\mathcal{S}} + \sum_{i=1}^n A_i \otimes h_i$ is positive. First assume $A_0 > 0$ is invertible. We claim

$$\begin{bmatrix} rA_0 \otimes 1_{\mathcal{S}} & \sum_{i=1}^n A_i \otimes h_i \\ \sum_{i=1}^n A_i^* \otimes h_i^* & rA_0 \otimes 1_{\mathcal{S}} \end{bmatrix}$$

is positive in $M_{2m} \otimes SOH(n)$, which is equivalent to

$$r^{-1} \left\| \sum_{i=1}^n A_0^{-1/2} A_i A_0^{-1/2} \otimes h_i \right\|_{M_m \otimes \mathcal{S}} \leq 1.$$

Write $B_i = A_0^{-1/2} A_i A_0^{-1/2}$, then by Proposition 3.3, $\|\sum_{i=1}^n B_i \otimes \overline{B_i}\| \leq 1$. Now embed $\mathcal{S} \subset B(\mathcal{H})$ and regard $h_i \otimes \overline{h_i}$ as an operator in $B(\mathcal{H} \otimes \overline{\mathcal{H}})$. Then by a version of Cauchy-Schwarz inequality due to Haagerup [4, Lemma 2.4],

$$\begin{aligned} r^{-1} \left\| \sum_{i=1}^n B_i \otimes h_i \right\|_{M_m \otimes \mathcal{S}} &\leq r^{-1} \left\| \sum_{i=1}^n B_i \otimes \overline{B_i} \right\|_{M_{2m}}^{1/2} \cdot \left\| \sum_{i=1}^n h_i \otimes \overline{h_i} \right\|_{B(\mathcal{H} \otimes \overline{\mathcal{H}})}^{1/2} \\ &\leq r^{-1} n^{1/2} < 1. \end{aligned}$$

Hence, the above matrix is positive as claimed. Pre and post multiplying it by $[1, 1]$ shows that $2(rA_0 \otimes 1_{\mathcal{S}} + \sum_{i=1}^n A_i \otimes h_i)$ is positive. When A_0 is not invertible, apply the standard $A_0 + \varepsilon I_m$ argument as in the proof of Proposition 3.3. Consequently, ϕ is completely positive. \square

Corollary 4.8. *In the previous settings, if \mathcal{S} is an operator system, then the map $\theta: \mathcal{S}^d \rightarrow SOH(n)$ by $\theta(f) = rf(1_{\mathcal{S}})H_0 + \sum_{i=1}^n f(h_i)H_i$ is completely positive.*

Proof. The dual map $\phi^d: \mathcal{S}^d \rightarrow SOH(n)^d$, $\phi^d(f)(H_i) = f \circ \phi(H_i)$, is completely positive. Let $\kappa: SOH(n) \rightarrow SOH(n)^d$ be the map $h_i \mapsto \delta_i$ as in Theorem 3.4. Then by self-duality of $SOH(n)$, the map $\kappa^{-1} \circ \phi^d: \mathcal{S}^d \rightarrow SOH(n)$ is completely positive and an easy calculation shows that $\kappa^{-1} \circ \phi^d(f) = \theta(f)$. \square

5. THE γ_{soh} -TENSOR PRODUCT

One of the important Banach space tensor products arises via factorization of bounded maps through Hilbert space. In this section and the next we construct a tensor product of two operator systems that arises from factorization of completely positive maps through SOH.

In [9], it is shown that the positive cone of the maximal tensor product of finite dimensional operator systems, $\mathcal{S} \otimes_{\max} \mathcal{T}$, can be identified with the completely positive maps from \mathcal{S}^d to \mathcal{T} that factor through M_n approximately; equivalently these are the nuclear maps. Motivated by this characterization, we will construct the γ_{soh} tensor product similarly by using $M_p(SOH(n))$ instead of M_n . We show that $\phi_1 \otimes \phi_2: \mathcal{S}_1 \otimes_{\gamma_{soh}} \mathcal{S}_2 \rightarrow \mathcal{T}_1 \otimes_{\gamma_{soh}} \mathcal{T}_2$ is completely positive whenever $\phi_i: \mathcal{S}_i \rightarrow \mathcal{T}_i$ is completely positive. We prove that γ_{soh} is a functorial and symmetric tensor product structure in the category of finite dimensional operator systems. We also prove that γ_{soh} is a distinct tensor product from many of the functorial tensors studied in [7, 8, 1].

Definition 5.1. *Let \mathcal{S} and \mathcal{T} be operator systems. We say that $\hat{u}: \mathcal{S}^d \rightarrow \mathcal{T}$ **factors through SOH approximately**, provided there exist nets of completely positive maps $\phi_\lambda: \mathcal{S}^d \rightarrow M_{p_\lambda}(SOH(n_\lambda))$ and $\psi_\lambda: M_{p_\lambda}(SOH(n_\lambda)) \rightarrow \mathcal{T}$ such that $\psi_\lambda \circ \phi_\lambda$ converges to \hat{u} in the point-norm topology.*

Definition 5.2 (The γ_{soh} -cone). *Let \mathcal{S} and \mathcal{T} be finite dimensional operator systems. Define*

$$\mathcal{C}_1^\gamma(\mathcal{S}, \mathcal{T}) := \{u \in \mathcal{S} \otimes \mathcal{T} : \hat{u} \text{ factors through SOH approximately}\}.$$

For $u = [u_{ij}] \in M_n(\mathcal{S} \otimes \mathcal{T})$, we regard $\hat{u} = [\hat{u}_{ij}]$ as a map from \mathcal{S}^d to $M_n(\mathcal{T})$. Thus there is no confusion to define $\mathcal{C}_n^\gamma(\mathcal{S}, \mathcal{T}) = \mathcal{C}_1^\gamma(\mathcal{S}, M_n(\mathcal{T}))$ in $M_n(\mathcal{S} \otimes \mathcal{T})$. We denote the triple $(\mathcal{S} \otimes \mathcal{T}, \{\mathcal{C}_n^\gamma(\mathcal{S}, \mathcal{T})\}_{n=1}^\infty, 1_{\mathcal{S}} \otimes 1_{\mathcal{T}})$ by $\mathcal{S} \otimes_{\gamma_{soh}} \mathcal{T}$.

Proposition 5.3. *The collection $\{\mathcal{C}_n^\gamma(\mathcal{S}, \mathcal{T})\}$ is a compatible family of proper cones of $\mathcal{S} \otimes \mathcal{T}$.*

Proof. Since $\mathcal{C}_n^\gamma(\mathcal{S}, \mathcal{T}) = \mathcal{C}_1^\gamma(\mathcal{S}, M_n(\mathcal{T}))$, it suffices to check that $\mathcal{C}_1^\gamma(\mathcal{S}, \mathcal{T})$ is a proper cone. It is obvious that $\mathcal{C}_1^\gamma(\mathcal{S}, \mathcal{T})$ is closed under positive scalar multiplication. Let $u_1, u_2 \in \mathcal{C}_1^\gamma(\mathcal{S}, \mathcal{T})$, so there are nets of completely positive maps $\phi_{\lambda_k}, \psi_{\lambda_k}$, where $k = 1, 2$ such that $\lim_\lambda \psi_{\lambda_k} \circ \phi_{\lambda_k} = \hat{u}_k$ in the point-norm topology.

Consider the directed set $\Lambda = \{(\lambda_1, \lambda_2)\}$ with the natural ordering. For each $\lambda = (\lambda_1, \lambda_2) \in \Lambda$, regard $M_{p_\lambda} = M_{p_{\lambda_1}} \oplus M_{p_{\lambda_2}}$ as the 2-by-2 block and let $n_\lambda = \max\{n_{\lambda_1}, n_{\lambda_2}\}$. Note that every completely positive map on $SOH(n_{\lambda_k})$, $k = 1, 2$, can be extended naturally on $SOH(n_\lambda)$. Thus without loss of generality we may assume that ϕ_{λ_k} maps into $M_{p_\lambda} \otimes SOH(n_\lambda)$ and ψ_{λ_k} has domain $M_{p_\lambda} \otimes SOH(n_{\lambda_k})$.

Thus, for each $\lambda = (\lambda_1, \lambda_2)$, we take $M_{p_\lambda}(SOH(n_\lambda))$, with completely positive maps $\phi_\lambda = \phi_{\lambda_1} \oplus \phi_{\lambda_2}$ and $\psi_\lambda = \psi_{\lambda_1} \oplus \psi_{\lambda_2}$. It remains to check

that $\psi_\lambda \circ \phi_\lambda$ converges to $\widehat{(u_1 + u_2)}$ in the point-norm topology. Indeed, given $f \in \mathcal{S}^d$ and $\varepsilon > 0$, by assumption there exist μ_1 and μ_2 so that $\|\hat{u}_k(f) - \psi_{\lambda_k} \circ \phi_{\lambda_k}(f)\| < \frac{\varepsilon}{2}$, for $\lambda_k > \mu_k$. Thus if $\mu = (\mu_1, \mu_2)$ and $\lambda > \mu$, then

$$\|\widehat{(u_1 + u_2)}(f) - (\psi_\lambda \circ \phi_\lambda)(f)\| \leq \sum_{k=1}^2 \|\hat{u}_k(f) - (\psi_k \circ \phi_k)(f)\| < \varepsilon$$

shows that $u_1 + u_2$ is in $\mathcal{C}_1^\gamma(\mathcal{S}, \mathcal{T})$.

Next we verify compatability. Let $u = [u_{ij}] \in \mathcal{C}_n^\gamma(\mathcal{S}, \mathcal{T})$ with \hat{u} factors through SOH approximately via nets ψ_λ and ϕ_λ . Write $A = [a_{kl}] \in M_{m,n}$, and write $w = AuA^* \in M_m(\mathcal{S} \otimes \mathcal{T})$. We claim that \hat{w} also factors through SOH approximately via the nets $(\theta_A \circ \psi_\lambda)$ and ϕ_λ , where $\theta_A: M_n(\mathcal{T}) \rightarrow M_m(\mathcal{T})$ by $B \mapsto ABA^*$ is completely positive. To this end, note that by writing $w = [\sum_{k,l=1}^n a_{i,k} u_{k,l} \overline{a_{l,j}}]_{i,j=1}^m$, for each $f \in \mathcal{S}^d$

$$\begin{aligned} \hat{w}(f) &= \left[\sum_{k,l=1}^n (a_{i,k} \hat{u}_{k,l} \overline{a_{l,j}})(f) \right]_{i,j=1}^m \\ &= \left[\sum_{k,l=1}^n a_{i,k} \hat{u}_{k,l}(f) \overline{a_{l,j}} \right]_{i,j=1}^m = A \hat{u}(f) A^* = (\theta_A \circ \hat{u})(f). \end{aligned}$$

Thus, for each $f \in \mathcal{S}^d$,

$$\|\hat{w}(f) - \theta_A \circ \psi_\lambda \circ \phi_\lambda(f)\| = \|\theta_A \circ (\hat{u} - \psi_\lambda \circ \phi_\lambda)(f)\| \rightarrow 0.$$

Therefore, $\{\mathcal{C}_n^\gamma(\mathcal{S}, \mathcal{T})\}$ is a compatible family of proper cones. \square

Proposition 5.4. *The unit $1_{\mathcal{S}} \otimes 1_{\mathcal{T}}$ is an Archimedean matrix order unit for $\mathcal{S} \otimes_{\gamma_{soh}} \mathcal{T}$.*

Proof. Again by identifying $\mathcal{C}_n^\gamma(\mathcal{S}, \mathcal{T}) = \mathcal{C}_1^\gamma(\mathcal{S}, M_n(\mathcal{T}))$, it suffices to prove that $1_{\mathcal{S}} \otimes 1_{\mathcal{T}}$ is an Archimedean order unit for $\mathcal{S} \otimes_{\gamma} \mathcal{T}$ on the ground level. Let $u \in \mathcal{S} \otimes \mathcal{T}$ be self-adjoint, we must find an $r > 0$ so that $r1_{\mathcal{S}} \otimes 1_{\mathcal{T}} - u$ is in $\mathcal{C}_1^\gamma(\mathcal{S}, \mathcal{T})$. Without loss of generality, we may assume $u = \sum_{i=1}^n x_i \otimes y_i$, where $x_i = x_i^*$ and $y_i = y_i^*$. By Proposition 4.6 and Corollary 4.7, there exist $r_1, r_2 > 0$ such that the map $\phi: \mathcal{S}^d \rightarrow SOH(n)$ by $\phi(f) = r_1 f(1_{\mathcal{S}}) H_0 - \sum_{i=1}^n f(x_i) H_i$, and $\psi: SOH(n) \rightarrow \mathcal{T}$ by $\psi(H_0) = r_2 1_{\mathcal{T}}$, $\psi(H_i) = y_i$ are completely positive. Choose $r = r_1 r_2$, then

$$\psi(\phi(f)) = r_1 r_2 f(1_{\mathcal{S}}) 1_{\mathcal{T}} - \sum_{i=1}^n f(x_i) y_i = (r1_{\mathcal{S}} \otimes 1_{\mathcal{T}} - u)(f)$$

shows that $(r1_{\mathcal{S}} \otimes 1_{\mathcal{T}} - u)$ factors through $SOH(n)$ exactly. Consequently, $1_{\mathcal{S}} \otimes 1_{\mathcal{T}}$ is an order unit for $\mathcal{S} \otimes_{\gamma} \mathcal{T}$.

Finally suppose $u = \sum_{i=0}^n x_i \otimes y_i \in \mathcal{S} \otimes \mathcal{T}$ and for each $\varepsilon > 0$, $u_\varepsilon = u + \varepsilon(1_{\mathcal{S}} \otimes 1_{\mathcal{T}}) \in \mathcal{C}_1^\gamma(\mathcal{S}, \mathcal{T})$. For each ε , there is a net of completely positive

maps $\phi_{\lambda_\varepsilon}$ and $\psi_{\lambda_\varepsilon}$ such that

$$\begin{array}{ccc} \mathcal{S}^d & \xrightarrow{\hat{u}_\varepsilon} & \mathcal{T} \\ & \searrow \phi_{\lambda_\varepsilon} \quad \nearrow \psi_{\lambda_\varepsilon} & \\ & M_{p_{\lambda_\varepsilon}}(SOH(n_{\lambda_\varepsilon})) & \end{array}$$

and $\|\hat{u}_\varepsilon(f) - (\psi_{\lambda_\varepsilon} \circ \phi_{\lambda_\varepsilon})(f)\| \rightarrow 0$, for each $f \in \mathcal{S}^d$.

Hence for each fixed ε , by finite dimensionality of \mathcal{S}^d , there exist a sufficiently large $k > \frac{1}{\varepsilon}$ and a pair of completely positive maps $\phi_{\lambda_{(\varepsilon,k)}}$ and $\psi_{\lambda_{(\varepsilon,k)}}$ from the net $(\psi_{\lambda_\varepsilon} \circ \phi_{\lambda_\varepsilon})$, such that $\|\hat{u}_\varepsilon(f) - (\psi_{\lambda_{(\varepsilon,k)}} \circ \phi_{\lambda_{(\varepsilon,k)}})(f)\| < \frac{1}{k}$, for every $\|f\| \leq 1$.

Consider the directed set Λ consisting of (ε, k) subject to the above condition, and order it by $(\varepsilon, k) \leq (\varepsilon', k')$ if and only if $\varepsilon' \leq \varepsilon$ and $k' \geq k$. Now we claim that $(\psi_\lambda \circ \phi_\lambda)_{\lambda \in \Lambda}$ converges to \hat{u} in the point-norm topology. Given $f \in \mathcal{S}^d$ with $\|f\| \leq 1$, for each $m > 0$, consider for $\varepsilon > \frac{1}{2m}$ and those $\lambda = (\varepsilon, k)$,

$$\begin{aligned} \|\hat{u}(f) - (\psi_\lambda \circ \phi_\lambda)(f)\| &= \|\hat{u}(f) - \hat{u}_\varepsilon(f) + \hat{u}_\varepsilon(f) - (\psi_\lambda \circ \phi_\lambda)(f)\| \\ &\leq \|\hat{u}(f) - \hat{u}_\varepsilon(f)\| + \|\hat{u}_\varepsilon(f) - (\psi_\lambda \circ \phi_\lambda)(f)\| \\ &< \frac{1}{2m} + \frac{1}{2m}. \end{aligned}$$

Therefore, \hat{u} factors through $M_p(SOH(n))$ approximately and $u \in \mathcal{C}_1^\gamma(\mathcal{S}, \mathcal{T})$. Consequently, $1_{\mathcal{S}} \otimes 1_{\mathcal{T}}$ is an Archimedean matrix order unit. \square

Definition 5.5. The triple $(\mathcal{S} \otimes \mathcal{T}, \mathcal{C}_n^\gamma(\mathcal{S}, \mathcal{T}), 1_{\mathcal{S}} \otimes 1_{\mathcal{T}})$ is an operator system, and we denote it by $\mathcal{S} \otimes_{\gamma_{soh}} \mathcal{T}$.

Theorem 5.6. The γ_{soh} -tensor defines a functorial operator system tensor product structure in the category of finite dimensional operator systems.

Proof. Let $P \in M_n(\mathcal{S})^+$ and $Q \in M_m(\mathcal{T})^+$. Note that by regarding $\mathcal{S} = \mathcal{S}^{dd}$ and $P: \mathcal{S}^d \rightarrow M_n$, then $(P \hat{\otimes} Q): \mathcal{S}^d \rightarrow M_{nm}(\mathcal{T})$ maps f to $P(f) \otimes Q$. Moreover, $P \hat{\otimes} Q$ factors through $M_n \otimes SOH(1)$ via

$$\begin{array}{ccc} \mathcal{S}^d & \xrightarrow{(P \hat{\otimes} Q)} & M_{nm}(\mathcal{T}) \\ & \searrow P \otimes H_0 \quad \nearrow I_n \otimes Q & \\ & M_n \otimes SOH(1) & \end{array}$$

Therefore, $P \otimes Q \in \mathcal{C}_{nm}^\gamma(\mathcal{S}, \mathcal{T})$.

For the functorial property, let $\rho: \mathcal{S} \rightarrow \mathcal{V}$ and $\kappa: \mathcal{T} \rightarrow \mathcal{W}$ be completely positive maps between finite dimensional operator systems, and let $u \in \mathcal{S} \otimes_{\gamma} \mathcal{T}$ be positive. Thus \hat{u} factors through $M_p(SOH(n))$ approximately

via some ϕ_λ and ψ_λ . Let $w = (\rho \otimes \kappa)(u) \in \mathcal{V} \otimes \mathcal{W}$. Notice this diagram

$$\begin{array}{ccc}
 \mathcal{V}^d & \xrightarrow{\hat{w}} & \mathcal{W} \\
 \rho^d \downarrow & & \uparrow \kappa \\
 \mathcal{S}^d & \xrightarrow{\hat{u}} & \mathcal{T} \\
 & \searrow \phi_\lambda \quad \nearrow \psi_\lambda & \\
 & M_{p_\lambda}(SOH(n_\lambda)) &
 \end{array}$$

commutes and the maps are all completely positive. Indeed, if $w = \sum_{i=1}^n \rho(x_i) \otimes \kappa(y_i)$, where $u = \sum_{i=1}^n x_i \otimes y_i$, then for each $f \in \mathcal{V}^d$,

$$\begin{aligned}
 \hat{w}(f) &= \sum_{i=1}^n f(\rho(x_i)) \kappa(y_i) = (\kappa \circ \hat{u} \circ \rho^d)(f) \\
 &= \lim_{\lambda} (\kappa \circ \psi_\lambda) \circ (\phi_\lambda \circ \rho^d)(f).
 \end{aligned}$$

Therefore, \hat{w} also factors through $M_p(SOH(n))$ approximately and $w \in (\mathcal{V} \otimes_{\gamma_{soh}} \mathcal{W})^+$. For $u = [u_{ij}] \in M_n(\mathcal{S} \otimes_{\gamma_{soh}} \mathcal{T})^+$, in the same vein we regard $\hat{u}: \mathcal{S} \rightarrow M_n(\mathcal{T})$. Then by replacing κ by $\kappa \otimes I_n$ and \mathcal{W} by $M_n(\mathcal{W})$ we deduce that \hat{u} factors through SOH approximately. Consequently $\rho \otimes \kappa$ is completely positive and the γ_{soh} -tensor product is functorial. \square

Remark 5.7. In [1], the *ess*-tensor product $\mathcal{S} \otimes_{ess} \mathcal{T}$ arises by the inclusion in $C_e^*(\mathcal{S}) \otimes_{\max} C_e^*(\mathcal{T})$, where $C_e^*(\mathcal{S})$ is the enveloping C^* -algebra of \mathcal{S} . It was yet to know whether this tensor product is functorial. Recently in [3, Proposition 3.2], it is shown that the *ess*-tensor product is not functorial. This allows us to distinguish γ_{soh} from *ess*.

Corollary 5.8. *The γ_{soh} -tensor product is not the *ess*-tensor product.*

We deduce further properties of the γ_{soh} -tensor product.

Proposition 5.9. *The γ_{soh} -tensor is symmetric.*

Proof. If $u \in (\mathcal{S} \otimes_{\gamma_{soh}} \mathcal{T})^+$, then by self-duality of $SOH(n)$ we see that

$$\begin{array}{ccc}
 \mathcal{T}^d & \xrightarrow{\hat{u}^d} & \mathcal{S}^{dd} = \mathcal{S} \\
 & \searrow \psi_\lambda^d \quad \nearrow \phi_\lambda^d & \\
 & M_{p_\lambda}(SOH(n_\lambda)) &
 \end{array}$$

commutes. Indeed, if $u = \sum_{i=1}^n x_i \otimes y_i$, then for $g \in \mathcal{T}^d$ and $f \in \mathcal{S}^d$,

$$(\hat{u}^d)(g)(f) = g(\hat{u}(f)) = \sum_{i=1}^n g(y_i) f(x_i) = \hat{u}(f)(g)$$

Hence \hat{u} admits an approximate factorization through $M_p(SOH(n))$ if and only if \hat{u}^d does. At the matrix level, we identify $M_n(\mathcal{S} \otimes_{\gamma_{soh}} \mathcal{T})^+ = (\mathcal{S} \otimes_{\gamma_{soh}}$

$M_n(\mathcal{T}))^+ = (M_n(\mathcal{T}) \otimes_{\gamma_{soh}} \mathcal{S})^+ = (\mathcal{T} \otimes_{\gamma_{soh}} M_n(\mathcal{S}))^+ = M_n(\mathcal{T} \otimes_{\gamma_{soh}} \mathcal{S})^+$. This shows that $x \otimes y \mapsto y \otimes x$ is a complete order isomorphism from $\mathcal{S} \otimes_{\gamma_{soh}} \mathcal{T}$ onto $\mathcal{T} \otimes_{\gamma_{soh}} \mathcal{S}$. \square

In [7], there are some tensor products constructed using the injective envelope. These come from the identifications, $\mathcal{S} \otimes_{el} \mathcal{T} \subset_{coi} \mathcal{I}(\mathcal{S}) \otimes_{\min} \mathcal{T}$, where $\mathcal{I}(\mathcal{S})$ is the injective envelope of \mathcal{S} , and likewise for $\mathcal{S} \otimes_{er} \mathcal{T}$. It turns out that the *el* and *er*-tensor products are not symmetric.

Corollary 5.10. *The γ_{soh} -tensor product is neither the *er* nor the *el*-tensor product.*

Theorem 5.11. *The γ_{soh} -tensor product is not the maximal tensor product. In particular, for $n \geq 2$, $SOH(n) \otimes_{\gamma_{soh}} SOH(n) \neq SOH(n) \otimes_{\max} SOH(n)$.*

Proof. By self-duality of $SOH(n)$, it suffices to show that $SOH(n)^d \otimes_{\gamma_{soh}} SOH(n) \neq SOH(n)^d \otimes_{\max} SOH(n)$. Consider the element $u = \sum_{i=0}^n \delta_i \otimes H_i$. Note that \hat{u} is in fact the identity map on $SOH(n)$ and factors through SOH trivially, so $u \in (SOH(n)^d \otimes_{\gamma_{soh}} SOH(n))^+$.

On the other hand, if $u \in SOH(n)^d \otimes_{\max} SOH(n)$ were positive, then by [9, Theorem 16], \hat{u} factors through the matrix algebras approximately. By a result of [5, Corollary 3.2], $SOH(n)$ must be (min, max)-nuclear and thus is unittally completely order isomorphic to a finite dimensional C^* -algebra. However it follows that $OH(n)$ could be completely isometrically represented on a finite dimensional Hilbert space and is hence 1-exact, contradicting Pisier's result [12]. Therefore, u is not positive in $SOH(n)^d \otimes_{\max} SOH(n)$. Consequently the two operator systems are not completely isomorphic. \square

We have seen that γ_{soh} is indeed a new tensor product. The next natural question is to ask which operator systems are nuclear with respect to γ_{soh} . The following result characterizes (min, γ_{soh})-nuclearity by identifying the matricial cone structures of the minimal tensor product to completely positive maps.

Theorem 5.12. *Let \mathcal{S} and \mathcal{T} be finite dimensional operator systems. Then $\mathcal{S} \otimes_{\min} \mathcal{T} = \mathcal{S} \otimes_{\gamma_{soh}} \mathcal{T}$ if and only if every completely positive map from \mathcal{S}^d to \mathcal{T} factors through SOH approximately.*

Proof. By [2, Proposition 1.9], $\mathcal{S} \otimes_{\min} \mathcal{T} =_{ucoi} (\mathcal{S}^d \otimes_{\max} \mathcal{T}^d)^d$, whose cone $(\mathcal{S}^d \otimes_{\max} \mathcal{T}^d)^{d,+}$ is in one-to-one correspondence to $CP(\mathcal{S}^d, \mathcal{T})$. Hence $\phi \in CP(\mathcal{S}^d, \mathcal{T})$ if and only if $\phi = \hat{u}$ for some $u \in (\mathcal{S} \otimes_{\min} \mathcal{T})^+$; and \hat{u} factors through SOH approximately if and only if $u \in (\mathcal{S} \otimes_{\gamma_{soh}} \mathcal{T})^+$. Consequently, $(\mathcal{S} \otimes_{\min} \mathcal{T})^+ = (\mathcal{S} \otimes_{\gamma_{soh}} \mathcal{T})^+$ if and only if every completely positive $\phi: \mathcal{S}^d \rightarrow \mathcal{T}$ admits such a factorization. At the matrix level, we identify $M_n(\mathcal{S} \otimes_{\tau} \mathcal{T})^+$ to $(\mathcal{S} \otimes_{\tau} M_n(\mathcal{T}))^+$ for $\tau = \min, \gamma_{soh}$; then the result follows from the base case. \square

Corollary 5.13. *$SOH(n)$ is (min, γ_{soh})-nuclear.*

Corollary 5.14. *The γ_{soh} -tensor product is not self-dual.*

Proof. Suppose γ_{soh} is self-dual; that is, $(\mathcal{S} \otimes_{\gamma_{soh}} \mathcal{T})^d = \mathcal{S}^d \otimes_{\gamma_{soh}} \mathcal{T}^d$. Then $SOH(n) \otimes_{\min} SOH(n)^d = SOH(n) \otimes_{\gamma_{soh}} SOH(n)^d$ and by dualizing one obtains $SOH(n)^d \otimes_{\max} SOH(n) = SOH(n)^d \otimes_{\gamma_{soh}} SOH(n)$, which is a contradiction. \square

6. EXTENSION TO INFINITE DIMENSIONAL OPERATOR SYSTEMS

In this section we show that every functorial tensor product structure defined on the category of finite dimensional operator systems can be extended to infinite dimensional operator systems. We also prove that this extension preserves symmetry, injectivity, and projectivity. Therefore, the γ_{soh} -tensor product defined in the previous section can now be extended to infinite dimensional operator systems.

Given an operator system \mathcal{S} , we denote the collection of finite dimensional operator subsystems of \mathcal{S} by $\mathcal{F}(\mathcal{S})$.

Definition 6.1. *Let τ be a functorial tensor product structure on the category of finite dimensional operator systems. We define $\tilde{\tau}$ on the category of operator systems in the following way: Given \mathcal{S} and \mathcal{T} , for each $n \in \mathbb{N}$, define the family of proper cones*

$$\mathcal{C}_n^{\tilde{\tau}}(\mathcal{S}, \mathcal{T}) := \bigcup_{E \in \mathcal{F}(\mathcal{S}), F \in \mathcal{F}(\mathcal{T})} M_n(E \otimes_{\tau} F)^+.$$

Theorem 6.2. *$\tilde{\tau}$ defines a functorial tensor product structure on the category of operator systems.*

Proof. Let us denote $\mathcal{C}_n^{\tilde{\tau}} = \mathcal{C}_n^{\tilde{\tau}}(\mathcal{S}, \mathcal{T})$. We first claim that it defines a matrix-ordering on $\mathcal{S} \otimes \mathcal{T}$. It is trivial that $\mathcal{C}_n^{\tilde{\tau}}$ is a proper cone for each n . To show that this is a matrix-ordering, we first check that for each $m, n \in \mathbb{N}$, $A \in M_{n,m}(\mathbb{C})$, $A^* \mathcal{C}_n^{\tilde{\tau}} A \subset \mathcal{C}_m^{\tilde{\tau}}$. Since every $B \in \mathcal{C}_n^{\tilde{\tau}}$ belongs to $M_n(E \otimes_{\tau} F)^+$, for some $E \in \mathcal{F}(\mathcal{S})$ and $F \in \mathcal{F}(\mathcal{T})$, we have $A^* B A \in \mathcal{C}_m(E \otimes_{\alpha} F) \subset \mathcal{C}_m^{\tilde{\tau}}$.

To see that $1 \otimes 1$ is an Archimedean matrix order unit for $(\mathcal{S} \otimes \mathcal{T}, \mathcal{C}_n^{\tilde{\tau}})$, consider $A \in M_n(\mathcal{S} \otimes \mathcal{T})$ such that for each $\varepsilon > 0$, $A_{\varepsilon} = \varepsilon(1 \otimes 1) \otimes I_n + A \in \mathcal{C}_n^{\tilde{\tau}}$. By definition, there exist $E_{\varepsilon} \in \mathcal{F}(\mathcal{S})$ and $F_{\varepsilon} \in \mathcal{F}(\mathcal{T})$ for which $A_{\varepsilon} \in M_n(E_{\varepsilon} \otimes_{\tau} F_{\varepsilon})^+$. Let $E = \bigcap_{\varepsilon > 0} E_{\varepsilon} \in \mathcal{F}(\mathcal{S})$ and $F = \bigcap_{\varepsilon > 0} F_{\varepsilon} \in \mathcal{F}(\mathcal{T})$, then by functorial property of τ , for each $\varepsilon > 0$, $M_n(E \otimes_{\tau} F)^+ \subsetneq M_n(E_{\varepsilon} \otimes_{\tau} F_{\varepsilon})^+$. Finally, since $E \otimes_{\tau} F$ defines an operator system, as $\varepsilon \rightarrow 0$ we see that $A_{\varepsilon} \rightarrow A \in M_n(E \otimes_{\tau} F)^+$. Consequently, $1 \otimes 1$ is an Archimedean matrix order unit; and $(\mathcal{S} \otimes \mathcal{T}, \mathcal{C}_n^{\tilde{\tau}}, 1 \otimes 1)$ is an operator system.

It remains to show the (T2) and (T3) properties. Given $P = (p_{ij}) \in M_n(\mathcal{S})^+$ and $Q = (q_{st}) \in M_m(\mathcal{T})^+$, by choosing E and F to be the spans of p_{ij} 's and q_{st} 's, we have $P \otimes Q \in M_{nm}(E \otimes_{\tau} F)^+$. This shows that $M_n(\mathcal{S})^+ \otimes M_m(\mathcal{T})^+ \subset \mathcal{C}_{nm}^{\tilde{\tau}}$. For (T3), we show further that it is functorial. Suppose $\phi: \mathcal{S}_1 \rightarrow \mathcal{S}_2$ and $\psi: \mathcal{T}_1 \rightarrow \mathcal{T}_2$ are completely positive maps. If $A \in \mathcal{C}_k^{\tilde{\tau}}$, then there are $E_1 \in \mathcal{F}(\mathcal{S})$ and $F_1 \in \mathcal{F}(\mathcal{T})$ such that $A \in M_k(E_1 \otimes_{\tau} F_1)^+$.

Let E_2 and F_2 denote the ranges of ϕ and ψ , respectively. By functorial property of τ , the map $\phi \otimes \psi|_{E_1 \otimes_\tau F_1}: E_1 \otimes_\tau F_1 \rightarrow E_2 \otimes_\tau F_2$ is completely positive. In particular, $(\phi \otimes \psi)^{(k)}(A) \in M_k(E_2 \otimes_\tau F_2)^+$. Therefore, $\phi \otimes \psi$ is completely positive and $\tilde{\tau}$ is functorial. \square

Proposition 6.3. *$\tilde{\tau}$ preserves injectivity, symmetry, and projectivity.*

Proof. Let τ be injective, $\mathcal{S}_1 \subset \mathcal{S}$ and $\mathcal{T}_1 \subset \mathcal{T}$ be operator subsystems, and $A \in M_n(\mathcal{S} \otimes_{\tilde{\tau}} \mathcal{T})^+ \cap M_n(\mathcal{S}_1 \otimes \mathcal{T}_1)$. By definition, $A \in M_n(E \otimes_\tau F)^+$ for some finite dimensional operator subsystems $E \subset \mathcal{S}$ and $F \subset \mathcal{T}$. Hence $E \cap \mathcal{S}_1$ and $F \cap \mathcal{T}_1$ are finite dimensional operator subsystems of \mathcal{S}_1 and \mathcal{T}_1 respectively. By injectivity of τ ,

$$A \in M_n(E \otimes_\tau F)^+ \cap M_n(\mathcal{S}_1 \otimes \mathcal{T}_1) = M_n((E \cap \mathcal{S}_1) \otimes_\tau (F \cap \mathcal{T}_1))^+.$$

This shows that $A \in M_n(\mathcal{S}_1 \otimes_{\tilde{\tau}} \mathcal{T}_1)^+$, and $\mathcal{S}_1 \otimes_{\tilde{\tau}} \mathcal{T}_1$ is complete order included in $\mathcal{S} \otimes_{\tilde{\tau}} \mathcal{T}$, proving $\tilde{\tau}$ is injective.

Let τ be symmetric, and $\phi: \mathcal{S} \otimes_{\tilde{\tau}} \mathcal{T} \rightarrow \mathcal{T} \otimes_{\tilde{\tau}} \mathcal{S}$ be the map $x \otimes y$ to $y \otimes x$. If $u \in M_n(\mathcal{S} \otimes_{\tilde{\tau}} \mathcal{T})^+$, then $u \in M_n(E \otimes_\tau F)^+$, for some finite dimensional E and F ; so $\phi^{(n)}(u) \in M_n(F \otimes_\tau E)^+ \subset M_n(\mathcal{S} \otimes_{\tilde{\tau}} \mathcal{T})^+$ and $\tilde{\tau}$ is symmetric.

Suppose τ is projective, and $q: \mathcal{S} \rightarrow \mathcal{V}$ and $\rho: \mathcal{T} \rightarrow \mathcal{W}$ are complete quotient maps. We claim that every $U \in M_n(\mathcal{V} \otimes_{\tilde{\tau}} \mathcal{W})^+$ can lift to a positive $\tilde{U} \in M_n(\mathcal{S} \otimes_{\tilde{\tau}} \mathcal{T})$. Since $U \in M_n(X \otimes_\tau Y)^+$, for some $X \in \mathcal{F}(\mathcal{V})$ and $Y \in \mathcal{W}(\mathcal{T})$, using projectivity of τ , there is $\tilde{U} \in M_n(E \otimes_\tau F)^+$ for which $E \in \mathcal{F}(\mathcal{S})$, $F \in \mathcal{F}(\mathcal{T})$ and $q \otimes \rho(\tilde{U}) = U$. Therefore, $\tilde{\tau}$ is projective. \square

Remark 6.4. *We remark that $\tilde{\tau}$ indeed extends τ . If \mathcal{S} and \mathcal{T} are finite dimensional, then $\mathcal{C}_n^{\tilde{\tau}}(\mathcal{S}, \mathcal{T}) = M_n(\mathcal{S} \otimes_\tau \mathcal{T})^+$ by functorial property of τ , thus $\mathcal{S} \otimes_\tau \mathcal{T} = \mathcal{S} \otimes_{\tilde{\tau}} \mathcal{T}$.*

Lemma 6.5. *Let τ be a symmetric tensor product structure. Then τ is left projective (resp. injective) if and only if it is right projective (resp. injective), if and only if it is projective (resp. injective).*

Proof. Let $q: \mathcal{S} \rightarrow \mathcal{R}$ be a complete quotient map. Then this commuting diagram

$$\begin{array}{ccc} \mathcal{S} \otimes_\tau \mathcal{T} & \longrightarrow & \mathcal{T} \otimes_\tau \mathcal{S} \\ q \otimes id \downarrow & & \downarrow id \otimes q \\ \mathcal{R} \otimes_\tau \mathcal{T} & \longleftarrow & \mathcal{T} \otimes_\tau \mathcal{R} \end{array}$$

asserts the equivalent condition. Similarly, if \mathcal{R} is a operator subsystem of \mathcal{S} , then

$$\begin{array}{ccc} \mathcal{S} \otimes_\tau \mathcal{T} & \longrightarrow & \mathcal{T} \otimes_\tau \mathcal{S} \\ \iota \otimes id \downarrow & & \downarrow id \otimes \iota \\ \mathcal{R} \otimes_\tau \mathcal{T} & \longleftarrow & \mathcal{T} \otimes_\tau \mathcal{R} \end{array}$$

shows that τ is left injective if and only if it is right injective. \square

Henceforth, given τ on finite dimensional operator systems, there is no ambiguity to say τ defines a tensor product structure on arbitrary operator systems. By this natural extension, we see that γ_{soh} defines a symmetric tensor product structure on operator systems. The cone $M_n(\mathcal{S} \otimes_{\gamma_{soh}} \mathcal{T})^+$ is precisely the set of $u \in \mathcal{S} \otimes M_n(\mathcal{T})$ so that $\hat{u}: E^d \rightarrow M_n(F)$ factors through SOH approximately, for some $E \in \mathcal{F}(\mathcal{S})$ and $F \in \mathcal{F}(\mathcal{T})$.

Some questions about γ_{soh} remain. We do not know if it is injective or projective. By the lemma above, it suffices to check these properties on one side. We do not yet know if γ_{soh} is distinct from the commuting tensor or any of the symmetric tensors that arise from two-sided inclusions into the maximal tensor products of the injective envelope or the C^* -envelope.

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REFERENCES

- [1] D. Farenick, A. S. Kavruk, V. I. Paulsen, and I. G. Todorov, *Operator systems from discrete groups*, Communications in Mathematical Physics **329** (2014), no. 1, 207–238.
- [2] D. Farenick and V. I. Paulsen, *Operator system quotients of matrix algebras and their tensor products*, arXiv preprint arXiv:1101.0790 (2011).
- [3] V. P. Gupta and P. Luthra, *Operator system nuclearity via C^* -envelopes*, arXiv preprint arXiv:1408.4312 (2015).
- [4] Uffe Haagerup, *Injectivity and decomposition of completely bounded maps*, Springer, 1985.
- [5] K. H. Han and V. I. Paulsen, *An approximation theorem for nuclear operator systems*, Journal of Functional Analysis **261** (2011), no. 4, 999–1009.
- [6] A. S. Kavruk, *Nuclearity related properties in operator systems*, arXiv preprint arXiv:1107.2133 (2011).
- [7] A. S. Kavruk, V. I. Paulsen, I. G. Todorov, and M. Tomforde, *Tensor products of operator systems*, Journal of Functional Analysis **261** (2011), no. 2, 267–299.
- [8] ———, *Quotients, exactness, and nuclearity in the operator system category*, Advances in Mathematics **235** (2013), 321–360.
- [9] W. H. Ng, *Two characterizations of the maximal tensor product of operator system*, arXiv preprint arXiv:1503.07097 (2015).
- [10] ———, *Completely bounded maps and operator algebras*, vol. 78, Cambridge University Press, 2002.
- [11] V. I. Paulsen, I. G. Todorov, and M. Tomforde, *Operator system structures on ordered spaces*, Proceedings of the London Mathematical Society (2010), pdq011.
- [12] G. Pisier, *The operator hilbert space OH , complex interpolation, and tensor norms*, vol. 122, American Mathematical Soc., 1996.
- [13] ———, *The operator hilbert space OH and type III von neumann algebras*, Bulletin of the London Mathematical Society **36** (2004), no. 4, 455–459.
- [14] Gerd Wittstock, *Ein operatorwertiger hahn-banach satz*, Journal of Functional Analysis **40** (1981), no. 2, 127–150.
- [15] C. Zhang, *Representation and geometry of operator spaces*, Ph.D. thesis, Univ. of Houston, 1995.

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